

TORUS EQUIVARIANT K-STABILITY

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ABSTRACT. We prove (using algebro-geometric methods) two results that allow to test the positivity of the Donaldson-Futaki weights of arbitrary polarised varieties via test-configurations which are equivariant with respect to a maximal torus in the automorphism group. It follows in particular that there is a purely algebro-geometric proof of the K-stability of projective space (or more generally of smooth toric Fanos with vanishing Futaki character, as well as of the examples of non-toric Kähler-Einstein Fano threefolds due to Ilten and Süß) and that K-stability for toric polarised manifolds can be tested via toric test-configurations. A further application is a proof of the K-stability of constant scalar curvature polarised manifolds with continuous automorphisms. Our approach is based on the method of filtrations introduced by Wyt Nystrom and Székelyhidi and indeed many of our results also extend to the class of (not necessarily finitely generated) polynomial filtrations.

1. INTRODUCTION

By the Yau-Tian-Donaldson conjecture for Fano manifolds [35, 33, 13], solved by Chen, Donaldson and Sun [6, 7, 8, 9], we know that a smooth Fano M admits a Kähler-Einstein metric if and only if it is K-stable, a purely algebro-geometric condition expressed through the positivity of certain GIT weights (known as Donaldson-Futaki weights or invariants).

A Kähler-Einstein metric, or, more generally, a constant scalar curvature Kähler metric, if it exists, can always be taken invariant under the action of a compact group of automorphisms of M . From the GIT point of view, when the point whose stability we would like to study has a non-trivial reductive stabiliser H , the Hilbert-Mumford Criterion can be strengthened: it is enough to consider one parameter subgroups which commute with H [36]. These facts suggest that there should be an equivariant version of the previous conjecture, both in the Fano and in the general case.

Recently Datar and Székelyhidi [11] have given a new proof of the Yau-Tian-Donaldson conjecture, showing the existence of Kähler-Einstein metrics along the smooth continuity method under the K-stability assumption. As a consequence they establish an equivariant version of the Chen-Donaldson-Sun theorem.

Theorem 1 (Székelyhidi-Datar [11] Theorem 1). *Fix a compact subgroup $G \subset \text{Aut}(M)$. Suppose that (M, K_M^{-1}) is K-stable with respect to G -equivariant special degenerations. Then M admits a Kähler-Einstein metric. In*

particular (M, K_M^{-1}) is K -stable with respect to all special degenerations (by the results of Berman [3]).

The relevant notions of special degenerations, test-configurations and K -stability are briefly recalled in Section 2 (see in particular Definitions 20 and 22).

The purpose of the present paper is to prove two algebro-geometric analogues of Theorem 1 and to discuss several applications. Our results have a more restrictive hypothesis on G - we assume that it is a maximal torus - but they hold for any polarised variety rather than just Fano manifolds. It should also be pointed out that the main application to Kähler-Einstein metrics discussed in [11] concerns the case of a maximal torus $T \subset \text{Aut}(M)$ of dimension $\dim(M)$ (the toric case) or $\dim(M) - 1$ (the complexity-one case).

We fix a polarised variety (X, L) , not necessarily Fano, and a test-configuration $(\mathcal{X}, \mathcal{L})$. We denote by $\text{DF}(\mathcal{X}, \mathcal{L})$ its Donaldson-Futaki invariant and by $\|(\mathcal{X}, \mathcal{L})\|_{L^2}$ its L^2 norm. We write $\text{Aut}(X, L)$ for the connected component of the group of biholomorphisms of X endowed with a lift to L .

Theorem 2. *Let $(\mathcal{X}, \mathcal{L})$ be a test-configuration for the polarised variety (X, L) . Fix a maximal torus $T \subset \text{Aut}(X, L)$. Then there exists a test-configuration $(\mathcal{X}', \mathcal{L}')$ for (X, L) such that*

- $(\mathcal{X}', \mathcal{L}')$ is T -equivariant,
- $\|(\mathcal{X}', \mathcal{L}')\|_{L^2} = \|(\mathcal{X}, \mathcal{L})\|_{L^2}$,
- $\text{DF}(\mathcal{X}', \mathcal{L}') = \text{DF}(\mathcal{X}, \mathcal{L})$,
- if the central fibre \mathcal{X}_0 is not isomorphic to X then the central fibre \mathcal{X}'_0 is not isomorphic to X .

Under stronger hypotheses on X and \mathcal{X} we can prove a more precise result.

Theorem 3 (Equivariant K -stability). *Let X be normal. Suppose that $(\mathcal{X}, \mathcal{L})$ is a test-configuration for (X, L) with normal total space \mathcal{X} and central fibre \mathcal{X}_0 not isomorphic to X . Let $T \subset \text{Aut}(X, L)$ be a maximal torus. Then there is a T -equivariant test-configuration $(\mathcal{X}', \mathcal{L}')$ with normal total space \mathcal{X}' such that $\text{DF}(\mathcal{X}', \mathcal{L}') \leq \text{DF}(\mathcal{X}, \mathcal{L})$ and with central fibre \mathcal{X}'_0 not isomorphic to X .*

In other words if (X, L) is K -stable with respect to T -equivariant test-configurations with normal total space which are not a product then it is K -stable (see Definition 22).

Remark. Note that quite often in the literature K -stability in the sense of Definition 22, in the presence of continuous automorphisms, is referred to as K -polystability (in order to be closer to the classical terminology in GIT).

Let us spell out the consequences of our result in the Fano case. Theorem 3 together with the equivariant version of the following result of Li and Xu gives a purely algebro-geometric proof of the K -stability part of Theorem 1.

Theorem 4 (Li-Xu [19]). *Suppose that $(X, -K_X)$ is a smooth Fano and let $(\mathcal{X}, \mathcal{L})$ be a test-configuration with normal total space \mathcal{X} and with $(\mathcal{X}_0, \mathcal{L}_0)$ not isomorphic to $(X, -K_X)$. Then there exists a special degeneration $(\tilde{\mathcal{X}}, \tilde{\mathcal{L}})$ with $\text{DF}(\tilde{\mathcal{X}}, \tilde{\mathcal{L}}) \leq \text{DF}(\mathcal{X}, \mathcal{L})$ and with $(\tilde{\mathcal{X}}_0, \tilde{\mathcal{L}}_0)$ not isomorphic to $(X, -K_X)$. Moreover if $(\mathcal{X}, \mathcal{L})$ is T -equivariant then we can choose the special degeneration $(\tilde{\mathcal{X}}, \tilde{\mathcal{L}})$ to be T -equivariant. (See [18] Remark 42 for a discussion of equivariance).*

Corollary 5. *Suppose that $(X, -K_X)$ is a smooth Fano and that it is K -stable with respect to T -equivariant special degenerations, $T \subset \text{Aut}(X, -K_X)$ denoting a maximal torus. Then $(X, -K_X)$ is K -stable.*

Proof. The result follows at once by applying Theorem 4 to the T -equivariant test-configuration $(\mathcal{X}', \mathcal{L}')$ of Theorem 3, with normal total space, obtained from a special degeneration $(\mathcal{X}, \mathcal{L})$. \square

Corollary 6. *There is a purely algebro-geometric proof that projective space \mathbb{P}^n is K -stable. More generally there is a purely algebro-geometric proof that smooth toric Fanos with vanishing Futaki character are K -stable. The same result holds for the examples of non-toric Kähler-Einstein Fano threefolds constructed by Ilten-Süss [16].*

Proof. As pointed out by Datar-Székelyhidi (in the proof of Theorem 17 of [11]) special degenerations of a toric manifold which are equivariant with respect to a maximal torus are necessarily product test-configurations, i.e. induced by a holomorphic vector field. By Corollary 5 it is enough to test K -stability with respect to such product test-configurations, which reduces to the vanishing of the classical Futaki character for holomorphic vector fields. Corollary 5 follows from Theorem 3 and the Li-Xu Theorem 4 which both have purely algebro-geometric proofs.

Essentially the same argument applies to the examples of non-toric Kähler-Einstein threefolds constructed in [16] Theorem 6.1. These are Fano threefolds of complexity one, i.e. the maximal torus $T \subset \text{Aut}(X, -K_X)$ has dimension $\dim(X) - 1$. In [16] a purely algebro-geometric proof is given that these are stable with respect to T -equivariant special degenerations. Combining this result with Corollary 5 gives an algebro-geometric proof of their K -stability. \square

Let us now go back to the general case of a not necessarily Fano polarised variety (X, L) . Theorem 3 applies in particular to toric polarised varieties. It seems worth pointing this out explicitly since normal toric test-configurations and their Donaldson-Futaki weight can be understood entirely in terms of rational piecewise linear convex functions on the moment polytope, see [13].

Corollary 7. *Let (X, L) be a (normal) toric polarised variety. If the Donaldson-Futaki invariant of (X, L) is positive on the class of nonproduct toric*

test-configurations with normal total space then it is positive on the class of all nonproduct test-configurations with normal total space.

Theorem 3 together with previous work of the second author and Székelyhidi [29] (based on [1, 14]) yield a result about the positivity of the Donaldson-Futaki weights of constant scalar curvature Kähler manifolds.

Theorem 8 (K-stability of cscK manifolds). *Suppose that (X, L) is a constant scalar curvature polarised manifold and $(\mathcal{X}, \mathcal{L})$ is a test-configuration with normal total space \mathcal{X} and with central fibre \mathcal{X}_0 not isomorphic to X . Then $\text{DF}(\mathcal{X}, \mathcal{L}) > 0$.*

When $\text{Aut}(X, L)$ (modulo the subgroup $\mathbb{C}^* \subset \text{Aut}(X, L)$ scaling the linearisation) is discrete Theorem 8 is the main result Theorem 1.2 of [27]. In the general case a very different proof of Theorem 8 has been given originally in a series of works by Mabuchi, see [20, 21]. When we had already completed the present work a paper by Berman-Darvas-Lu appeared [4], which also contains a new proof of Theorem 8, using completely different methods.

Remark. The proof of Theorem 1.2 in [27] requires that the normalisation of \mathcal{X} is not a trivial test-configuration. This assumption is mistakenly missing in [27]. See [19] Remark 4 and the note [28] for further discussion.

A key ingredient in our work is a procedure, introduced in Section 3, for specialising an arbitrary test-configuration to a T -equivariant one. This procedure relies on the approach to test-configurations via filtrations of the homogeneous coordinate ring introduced by Witt Nyström [34] and Székelyhidi [32] and further developed in [10]. The construction involves naturally a class of filtrations which in general are not finitely generated. An example of this phenomenon is discussed in the Appendix. These filtrations, even if they are not finitely generated, are polynomial in the sense of Definition 18. The notion of polynomial filtration has been introduced in [10].

Theorem 2 and Theorem 3 show in particular that if we start with a (normal) test-configuration $(\mathcal{X}, \mathcal{L})$ which is not induced by a holomorphic vector field on X then the corresponding (normal) T -equivariant test-configuration $(\mathcal{X}', \mathcal{L}')$ (for a *maximal* torus $T \subset \text{Aut}(X, L)$) is also not induced by a holomorphic vector field. We can prove results with a similar flavour under a different non-degeneracy condition, which is closer to the point of view of [32] based on the L^2 norm of test-configurations. We believe that this different non-degeneracy condition based on norms may be more fundamental.

Recall that Székelyhidi [31] defined a formal L^2 scalar product $\langle \alpha, \beta \rangle$ between \mathbb{C}^* -actions α, β on a polarised scheme. This can be used to define a scalar product between a one parameter subgroup β of $\text{Aut}(X, L)$ and a β -equivariant test-configuration. In Section 5 we extend this notion to a scalar product between two arbitrary test-configurations, which we denote by

$$\langle (\mathcal{X}, \mathcal{L}), (\mathcal{X}', \mathcal{L}') \rangle.$$

This scalar product can be extended further to filtrations. We should warn the reader that in general it does not make sense to ask about its bilinearity.

Given a torus T in $\text{Aut}(X, L)$, using this scalar product we can define the L^2 projection $(\mathcal{X}, \mathcal{L})_T$ along T of a test-configuration $(\mathcal{X}, \mathcal{L})$. If moreover $(\mathcal{X}, \mathcal{L})$ is also T equivariant (but not in general) we have a natural notion of orthogonal complement $(\mathcal{X}, \mathcal{L})_T^\perp$.

We say that a test-configuration is degenerate with respect to a maximal torus T if

$$\|(\mathcal{X}, \mathcal{L})\|_{L^2} = \|(\mathcal{X}, \mathcal{L})_T\|_{L^2}.$$

When $(\mathcal{X}, \mathcal{L})$ is T -equivariant this condition is equivalent to $\|(\mathcal{X}, \mathcal{L})_T^\perp\|_{L^2} = 0$. Remark that when T is trivial this non-degeneracy condition reduces to the non-degeneracy condition $\|(\mathcal{X}, \mathcal{L})\|_{L^2} > 0$ considered in [32].

In Proposition 43 we will show that the degeneracy condition does not depend on T , that is if a test-configuration is degenerate with respect to a maximal torus then it is degenerate with respect to all maximal tori. We can prove the analogue of Theorem 3 in the context of this alternative L^2 non-degeneracy condition.

Theorem 9 (Equivariant K-stability, L^2 version). *Let $(\mathcal{X}, \mathcal{L})$ be a test-configuration for the polarised variety (X, L) . Fix a maximal torus $T \subset \text{Aut}(X, L)$. Then there exists a test-configuration $(\mathcal{X}', \mathcal{L}')$ for (X, L) such that*

- $(\mathcal{X}', \mathcal{L}')$ is T -equivariant,
- $\|(\mathcal{X}', \mathcal{L}')\|_{L^2} = \|(\mathcal{X}, \mathcal{L})\|_{L^2}$,
- $\text{DF}(\mathcal{X}', \mathcal{L}') = \text{DF}(\mathcal{X}, \mathcal{L})$,
- $\|(\mathcal{X}', \mathcal{L}')_T\|_{L^2} = \|(\mathcal{X}, \mathcal{L})_T\|_{L^2}$. Thus if $\|(\mathcal{X}, \mathcal{L})_T\|_{L^2} < \|(\mathcal{X}, \mathcal{L})\|_{L^2}$ then $\|(\mathcal{X}', \mathcal{L}')_T^\perp\|_{L^2} > 0$, the latter denoting the usual L^2 orthogonal to the torus as in [31].

It follows that if (X, L) is K-stable with respect to all T -equivariant test-configurations with nontrivial L^2 orthogonal to T then it is K-stable with respect to all test-configurations which are not degenerate with respect to T .

Remark. One may define a notion of uniform K-stability with respect to a maximal torus $T \subset \text{Aut}(X, L)$ by requiring that for all test-configurations, not necessarily T -equivariant, we have

$$\text{DF}(\mathcal{X}, \mathcal{L}) \geq c(\|(\mathcal{X}, \mathcal{L})\|_{L^2}^2 - \|(\mathcal{X}, \mathcal{L})_T\|_{L^2}^2)^{\frac{1}{2}}$$

for some fixed $c > 0$. Theorem 9 then shows that this inequality holds if and only if it holds for T -equivariant test-configurations, in which case it is precisely the notion of uniform stability with respect to the L^2 norm introduced in [30] section 3.1.1. In particular it is known that it is only applicable to the two-dimensional case [5].

Similarly we can prove the analogue of Theorem 8 using only the L^2 nondegeneracy condition. The proof depends on [29] as well as on a result of Dervan [12] and of Boucksom-Hisamoto-Jonsson [5].

Theorem 10 (K-stability of cscK manifolds, L^2 version). *Let (X, L) , $(\mathcal{X}, \mathcal{L})$ be a constant scalar curvature polarised manifold and a test-configuration. Suppose that there exists a maximal torus $T \subset \text{Aut}(X, L)$ (not necessarily making $(\mathcal{X}, \mathcal{L})$ T -equivariant) for which $\|(\mathcal{X}, \mathcal{L})_T\|_{L^2} < \|(\mathcal{X}, \mathcal{L})\|_{L^2}$. Then $\text{DF}(\mathcal{X}, \mathcal{L}) > 0$. In other words $(\mathcal{X}, \mathcal{L})$ is K -stable with respect to all test-configurations which are not degenerate with respect to maximal tori.*

Remark. It is important to emphasise that $(\mathcal{X}, \mathcal{L})_T$ does coincide with the usual L^2 projection along T of [31] for T -equivariant test-configurations, but is otherwise something quite different. In particular if ξ is a holomorphic vector field on X which does not commute with T and generates a product test-configuration $(\mathcal{X}, \mathcal{L})$, then $(\mathcal{X}, \mathcal{L})_T$ is *not* the product test-configuration generated by the projection of ξ along T with respect to the Futaki-Mabuchi scalar product of vector fields (see [31] section 2). This will be clear from Definition 36. So Theorem 10 does not contradict the statement that the classical Futaki character of a constant scalar curvature Kähler manifold should vanish: in fact by Proposition 43 a product test-configuration is degenerate with respect to all maximal tori.

In Proposition 44 we show, independently of Theorem 10, that for test-configurations with normal total space the L^2 degeneracy condition is equivalent to being a product. Note that by the result of Dervan and of Boucksom-Hisamoto-Jonsson mentioned above it is known that $\|(\mathcal{X}, \mathcal{L})\|_{L^2} > 0$ holds if and only if the normalisation of $(\mathcal{X}, \mathcal{L})$ is trivial.

There are at least three good reasons for introducing the L^2 non-degeneracy condition. The first is that it appears naturally in the proof of Theorem 8; so naturally that we first prove Theorem 10 and then we explain how essentially the same proof implies also Theorem 8. Secondly the L^2 non-degeneracy condition generalises readily to the setting of not necessarily finitely generated filtrations, a point of view which we explain at the end of the present section.

The third reason is that the bilinear pairing used to define this condition gives a natural pseudo-metric on the space of test-configurations. We will explain this in Section 6 and only briefly sketch the construction here. Let Δ_r be the set of test-configurations for (X, L) of fixed exponent r , modulo base-change $z \mapsto z^p$ on the base \mathbb{C} and constant rescalings of the linearisation. As pointed out by Odaka [23] this space is in a natural bijection with the rational points of a simplicial complex, introduced classically in the theory of reductive algebraic groups, the Tits spherical building (i.e. Mumford's flag complex) of the group $GL(H^0(X, L^r)^\vee)$. The main idea behind this correspondence is that exponent r test-configurations are induced by one parameter subgroups of $GL(H^0(X, L^r)^\vee)$. The geometric realisation of this simplicial complex $\Delta_r = |\Delta(GL(H^0(X, L^r)^\vee))|_{\mathbb{R}}$ carries a complete metric ρ_r introduced by Tits. This metric is essentially the angular distance between one parameter subgroups of $GL(H^0(X, L^r)^\vee)$, the angle being defined by the Killing form.

There are natural maps $\iota_{r,k} : \Delta_r \rightarrow \Delta_{rk}$ mapping the class of a test-configuration $(\mathcal{X}, \mathcal{L})$ of exponent r to that of $(\mathcal{X}, \mathcal{L}^k)$ of exponent rk . Following Odaka [23] Corollary 2.4 one can define the space of test-configurations modulo tensor powers of the polarisation as the limit $\Delta = \lim_{\rightarrow, r} \Delta_r$. We introduce a pseudo-metric ρ on this space as

$$\rho((\mathcal{X}, \mathcal{L}), (\mathcal{X}', \mathcal{L}')) = \liminf_{k \rightarrow \infty} \rho_{krr'}(\iota_{r,kr'}(\mathcal{X}, \mathcal{L}), \iota_{r',kr}(\mathcal{X}', \mathcal{L}')),$$

writing r for the exponent of $(\mathcal{X}, \mathcal{L})$ and r' for the exponent of $(\mathcal{X}', \mathcal{L}')$. Note that by definition there is a uniform bound $\text{diam}(\Delta_r) \leq \pi$ and so $\rho((\mathcal{X}, \mathcal{L}), (\mathcal{X}', \mathcal{L}'))$ is always finite. There is a simple relation between this metric and the scalar product of test-configurations.

Proposition 11. *Suppose $(\mathcal{X}, \mathcal{L})$ and $(\mathcal{X}', \mathcal{L}')$ are two test-configurations with positive L^2 norm, then we have*

$$\rho((\mathcal{X}, \mathcal{L}), (\mathcal{X}', \mathcal{L}')) = \arccos \frac{\langle (\mathcal{X}, \mathcal{L}), (\mathcal{X}', \mathcal{L}') \rangle}{\|(\mathcal{X}, \mathcal{L})\|_{L^2} \|(\mathcal{X}', \mathcal{L}')\|_{L^2}}.$$

In other words our scalar product is a limit of the pairings induced by the Killing form.

There are of course alternative approaches for introducing a geometric structure on the space of test-configurations; two recent contributions are [5, 17]. It seems natural to ask about the relation between Δ - or rather its Kolmogorov quotient with respect to the pseudo-metric defined above - and the space constructed in [5] section 5.

For polarised manifolds (X, L) Odaka [23] suggests that Δ should be viewed as an algebraic counterpart to the space of geodesics on the space of Kähler metrics in the class $c_1(L)$. There is a vast literature on geometric structures on this space and we will not attempt to give references here.

Following [32] the norm, Donaldson-Futaki invariant and L^2 product extend to more general, not necessarily finitely generated filtrations. Theorem 9 extends readily to the class of not necessarily finitely generated, polynomial filtrations studied in [10].

Theorem 12. *Let χ denote a not necessarily finitely generated polynomial filtration of the homogeneous coordinate ring R of (X, L) (see Definition 18). Let $T \subset \text{Aut}(X, L)$ be a torus. Denote by χ_T the L^2 projection of χ along T . Then there exists a polynomial filtration χ' of R such that*

- χ' is preserved by the torus T acting on R ,
- $\|\chi'\|_{L^2, \text{pol}} = \|\chi\|_{L^2, \text{pol}}$,
- $\text{DF}_{\text{pol}}(\chi') = \text{DF}_{\text{pol}}(\chi)$,
- $\|\chi'_T\|_{L^2, \text{pol}} = \|\chi_T\|_{L^2, \text{pol}}$, so if $\|\chi_T\|_{L^2, \text{pol}} < \|\chi\|_{L^2, \text{pol}}$ then we also have $\|\chi'_T\|_{L^2, \text{pol}} < \|\chi'\|_{L^2, \text{pol}}$.

We do not know how to extend this result to non-polynomial filtrations at present. The notion of polynomial Donaldson-Futaki invariant DF_{pol} and polynomial norm $\|\cdot\|_{L^2, \text{pol}}$ of a polynomial filtration are recalled in

Definition 19; it is not clear to us if, in the non-finitely generated case, they are equal to the notions defined in [32].

The analogue of Theorem 8 in this context is not known and we propose it as a conjecture.

Conjecture 13. *Suppose (X, L) is a constant scalar curvature polarised manifold. Let χ denote a not necessarily finitely generated filtration of the homogeneous coordinate ring R of (X, L) (following the conventions of [32], see Definition 14) and let $T \subset \text{Aut}(X, L)$ be a maximal torus. Denote by χ_T the L^2 projection of χ along T . If $\|\chi_T\|_{L^2} < \|\chi\|_{L^2}$ then $\text{DF}(\chi) > 0$.*

Note that when $\text{Aut}(X, L)$ (modulo the subgroup $\mathbb{C}^* \subset \text{Aut}(X, L)$ scaling the linearisation) is discrete we have $\|\chi_T\|_{L^2} = 0$ and the nondegeneracy condition becomes $\|\chi\|_{L^2} > 0$. In this case Conjecture 13 is known and is the main result in [32].

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2. FILTRATIONS, TEST-CONFIGURATIONS, APPROXIMATIONS

Let (X, L) be a polarised variety. We consider its coordinate ring

$$R = R(X, L) = \bigoplus_{k \geq 0} R_k = \bigoplus_{k \geq 0} H^0(X, L^k).$$

To this ring we can attach the *Hilbert function*

$$h(k) = \dim R_k.$$

Since R is finitely generated the Hilbert function is actually a polynomial for all sufficiently large k . This polynomial, called the *Hilbert polynomial*, is of degree n , where $n = \dim X$. Recall that, up to rescaling L , we can assume that R is generated by R_1 . We are going to take a similar point of view on test-configurations and on the Donaldson-Futaki invariant.

Definition 14 (Filtrations [32]). *We define a filtration χ of R to be sequence of vector subspaces*

$$H^0(X, \mathcal{O}) = F_0 R \subset F_1 R \subset \dots$$

which is

- (i) exhaustive: for every k there exists a $j = j(k)$ such that $F_j R_k = H^0(X, L^k)$,
- (ii) multiplicative: $(F_i R_l)(F_j R_m) \subset F_{i+j} R_{l+m}$,
- (iii) homogeneous: $f \in F_i R$ then each homogeneous piece of f is in $F_i R$.

We can construct two algebraic objects out of a filtration.

Definition 15 (Rees and graded algebra). *Let χ be a filtration. The corresponding Rees algebra is*

$$\text{Rees}(\chi) = \bigoplus_{i \geq 0} F_i R t^i$$

The graded modules are

$$\text{gr}_i(H^0(X, L^k)) = F_i(H^0(X, L^k)) / F_{i-1}(H^0(X, L^k))$$

The graded algebra is

$$\text{gr}(\chi) = \bigoplus_{k, i \geq 0} \text{gr}_i(H^0(X, L^k))$$

The Rees algebra is a subalgebra of $R[t]$, and by the following elementary result it is possible to reconstruct χ from it.

Lemma 16 (Reconstruction Lemma). *Let A be a \mathbb{C} -subalgebra of $R[t]$. We define a filtration χ_A of R as follows*

$$F_i R = \{s \in R \mid t^i s \in A\}$$

The filtration χ_A satisfies the hypotheses of Definition 14 if and only if A satisfies the conditions

- $A \cap R = H^0(X, \mathcal{O}_X)$;
- for every $s \in H^0(X, L)$ there exists an i such that $t^i s \in A$;
- if $t^i f$ is in A , then, for each of the homogenous component f_k of f , $t^i f_k$ is also in A .

Moreover any filtration χ equals χ_A , where A is the Rees algebra of χ .

Given a filtration we can define the corresponding weight functions.

Definition 17 (Weight functions). *The weight function of χ is*

$$w_\chi(k) = w(k) = \sum_i (-i) \dim \text{gr}_i(H^0(X, L^k))$$

the trace squared function is

$$d_\chi(k) = d(k) = \sum_i i^2 \dim \text{gr}_i(H^0(X, L^k)).$$

There is no polynomiality result in this generality, hence we give the following definition to single out classes of filtrations which are easier to study.

Definition 18 (Finitely generated and polynomial filtrations). *A filtration is called finitely generated if its Rees algebra is finitely generated. A filtration is called polynomial if, for k large enough, $w(k)$ is a polynomial of degree $n + 1$ and $d(k)$ is a polynomial of degree $n + 2$.*

For a polynomial filtration there are expansions

$$\begin{aligned} h(k) &= a_0 k^n + a_1 k^{n-1} + \dots \\ w(k) &= b_0 k^{n+1} + b_1 k^n + \dots \\ d(k) &= c_0 k^{n+2} + c_1 k^{n+1} + \dots \end{aligned}$$

Definition 19 (Polynomial Donaldson-Futaki invariant and norm). *Let χ be a polynomial filtration. We define*

$$\begin{aligned} \text{DF}_{\text{pol}}(\chi) &= \frac{a_1 b_0 - a_0 b_1}{a_0^2} \\ \|\chi\|_{L^2, \text{pol}}^2 &= c_0 - \frac{b_0^2}{a_0}. \end{aligned}$$

One of the main objects of study in this paper are test-configurations. Let us briefly recall their definition.

Definition 20 (Test-configuration). *Let \mathbb{C}^* act in the standard way on \mathbb{C} . An exponent r test-configuration $(\mathcal{X}, \mathcal{L})$ for (X, L) is a \mathbb{C}^* -equivariant flat morphism $\pi: \mathcal{X} \rightarrow \mathbb{C}$, together with a π -very ample line bundle \mathcal{L} and a linearization of the action of \mathbb{C}^* to \mathcal{L} , such that the fibre over 1 is isomorphic to (X, \mathcal{L}^r) . We say that $(\mathcal{X}, \mathcal{L})$ is*

- *a product, if it is isomorphic to $(X \times \mathbb{C}, L^r \boxtimes \mathcal{O}_{\mathbb{C}})$, where the action of \mathbb{C}^* on $X \times \mathbb{C}$ is induced by a one parameter subgroup λ of $\text{Aut}(X, L)$ by $\lambda(\tau) \cdot (x, z) = (\lambda(\tau) \cdot x, \tau z)$;*
- *trivial, if it is a product and, moreover, the action of \mathbb{C}^* on X is trivial;*
- *normal, if the total space \mathcal{X} is normal;*
- *equivariant with respect to a subgroup H of $\text{Aut}(X, L)$, if the action of \mathbb{C}^* can be extended to an action of $\mathbb{C}^* \times H$ such that the action of $\{1\} \times H$ is the natural action of H on (X, L^r) ;*
- *in the Fano case, a test-configuration is special, if \mathcal{X} is normal, all the fibres are klt and a positive rational multiple of \mathcal{L} equals $-K_{\mathcal{X}}$ (this notion is due to Tian [33], see also [19] Definition 1).*

The normalisation of a test-configuration is the normalisation of \mathcal{X} endowed with the natural induced line bundle and \mathbb{C}^* action (or $\mathbb{C}^* \times H$ action). A test-configuration is a product if and only if the central fibre \mathcal{X}_0 is isomorphic to X : by standard theory in this case there is a trivialisation $\mathcal{X} \cong X \times \mathbb{C}$ and the \mathbb{C}^* -action on \mathcal{X} corresponds to a \mathbb{C}^* -action on $X \times \mathbb{C}$ preserving $X \times \{0\}$, which must then be induced by a \mathbb{C}^* -action on X as above.

Let us recall the relation between finitely generated filtrations and test-configurations, as developed by Witt Nyström [34] and Székelyhidi [32]. Let χ be a finitely generated filtration. The Rees algebra $\text{Rees}(\chi)$ is a finitely generated flat $\mathbb{C}[t]$ -module; this means that the associated relative Proj with its natural $\mathcal{O}(1)$ is a test-configuration $(\mathcal{X}, \mathcal{L})$. The central fibre is the Proj of the graded algebra $\text{gr}(\chi)$; the \mathbb{C}^* -action on the central fibre is given by minus the i -grading of $\text{gr}(\chi)$.

Theorem 21 ([32] section 3.1). *All test-configurations arise from finitely generated filtrations via the construction discussed above. This construction preserves the weight functions $w(k)$ and $d(k)$. In particular, finitely generated filtrations are polynomial and in this case Definition 19 agrees with the usual notions of the Donaldson-Futaki invariant and of the norm of a test-configuration.*

In view of this theorem we will feel free to write $\text{DF}(\chi)$ and $\|\chi\|_{L^2}$ rather than $\text{DF}_{\text{pol}}(\chi)$ and $\|\chi\|_{L^2, \text{pol}}$ for a finitely generated filtration. Note that implicitly we have defined also the Donaldson-Futaki invariant of a test-configuration.

Definition 22 (K-(semi)stability). *We say that a polarised variety (X, L) is K-(semi)stable with respect to a certain fixed class of test-configurations if we have $\text{DF}(\mathcal{X}, \mathcal{L}) \geq 0$ (respectively $\text{DF}(\mathcal{X}, \mathcal{L}) > 0$) for all the test-configurations belonging to the given class.*

We say (X, L) is K-stable (without further specification) if it is K-stable with respect to test-configurations with normal total space which are not a product.

Remark. It is useful to allow rational multiples of \mathbb{C}^* -actions in the definition of a test-configuration and of the relevant weights. For this we use the relation

$$\text{DF}(\mathcal{X}, \mathcal{L}, r\alpha) = r \text{DF}(\mathcal{X}, \mathcal{L}, \alpha).$$

for a test-configuration $(\mathcal{X}, \mathcal{L})$ with \mathbb{C}^* -action α .

In [32] a general definition of the Donaldson-Futaki invariant for non-finitely generated filtrations is given in terms of a specific sequence of finitely generated approximations. We do not know if the definition in [32] agrees with Definition 19 in the case of polynomial non-finitely generated filtrations.

Following [10, Section 2.3] we introduce a way to approximate non-finitely generated filtrations which is adapted to the case of polynomial filtrations. Let us point out that these approximations are different from the ones used in [32].

Definition 23 (Approximating filtrations). *Let χ be a filtration. We define $\text{Rees}^{(r)}(\chi)$ as the subalgebra of $R[t]$ generated by*

$$\bigoplus_{k \leq r, i \geq 0} F_i H^0(X, L^k) t^i$$

and we let $\chi^{(r)}$ be the filtration corresponding to $\text{Rees}^{(r)}(\chi)$ via Lemma 16.

Note that the filtration $\chi^{(r)}$ is finitely generated by construction.

Proposition 24. *Let χ be a polynomial filtration. Then for r large enough the filtrations $\chi^{(r)}$ have the same weight functions as χ and*

$$\mathrm{DF}_{\mathrm{pol}}(\chi) = \mathrm{DF}(\chi^{(r)}), \quad \|\chi\|_{L^2, \mathrm{pol}} = \|\chi^{(r)}\|_{L^2}.$$

Proof. The weight functions are polynomials, so it is enough to check that they agree on a large enough finite set of values of k . The filtrations χ and $\chi^{(r)}$ agree on $\bigoplus_{k \leq r} R_k$, hence the claim. \square

The group $\mathrm{Aut}(X, L)$ acts on R . We extend its action to $R[t]$ by letting it act trivially on t .

Definition 25 (Equivariant filtrations). *Let H be a subgroup of $\mathrm{Aut}(X, L)$. A filtration χ is H -equivariant if each $F_i R$ is an H -submodule of R .*

The following Lemma is elementary.

Lemma 26. *A filtration χ is H -equivariant if and only if $\mathrm{Rees}(\chi)$ is an H -submodule of $R[t]$. Moreover, if there exists a vector subspace V of $\mathrm{Rees}(\chi)$ which is preserved by H and generates $\mathrm{Rees}(\chi)$, then χ is H -equivariant.*

Lemma 27. *Let H be a subgroup of $\mathrm{Aut}(X, L)$ and χ be an H -equivariant filtration. Then the approximating filtrations $\chi^{(r)}$ of Definition 23 are H -equivariant.*

Proof. Apply Lemma 26 with the choice $V = \bigoplus_{k \leq r, i \geq 0} F_i H^0(X, L^k) t^i$. \square

The relation between equivariant test-configurations and equivariant filtrations is stated in the following proposition.

Proposition 28. *Let H be a subgroup of $\mathrm{Aut}(X, L)$. A test-configuration is H -equivariant if and only if it is given by an H -equivariant filtration.*

Proof. If a finitely generated filtration is H -equivariant there is a natural action of H on the Rees algebra which respects both the t and the i grading. So the corresponding test-configuration has an action of H which commutes with the action of \mathbb{C}^* . The converse follows from the explicit construction proving Theorem 21 given in [32] section 3.1. \square

3. SPECIALISATION OF A TEST-CONFIGURATION

In the classical situation of a torus T acting on a projective variety one can specialise a point p to a fixed point \bar{p} for the action of T . One picks a generic one parameter subgroup λ of T and the specialisation is $\bar{p} = \lim_{\tau \rightarrow 0} \lambda(t) \cdot p$. This specialisation does depend on λ and when we need to emphasise this dependence we will denote it by p_λ .

In this section we study specialisation under a torus action for test-configurations. An important point is that we do this using filtrations, without embedding the test-configuration in a fixed projective space.

Definition 29 (Specialisation of a filtration). *Let T be a torus in $\text{Aut}(X, L)$ and λ a generic one parameter subgroup of T . Let χ be a filtration in the sense of Definition 14. We define the specialisation $\bar{\chi} = \chi_\lambda$ as*

$$\bar{F}_i H^0(X, L^k) = \lim_{\tau \rightarrow 0} \lambda(\tau) \cdot F_i H^0(X, L^k)$$

where we think of $F_i H^0(X, L^k)$ as a point in the relevant Grassmanian.

The following result collects the main properties of the specialisation $\bar{\chi}$.

Theorem 30. *The filtration $\bar{\chi}$ constructed in Definition 29 is a filtration in the sense of Definition 14 and is T -equivariant. Moreover the weight and trace-squared functions of $\bar{\chi}$ are equal to the weight and trace-squared functions of χ . In particular if χ is polynomial the same is true for $\bar{\chi}$ and we have*

$$DF(\chi)_{\text{pol}} = DF(\bar{\chi})_{\text{pol}}, \quad \|\chi\|_{L^2, \text{pol}} = \|\bar{\chi}\|_{L^2, \text{pol}}.$$

Proof. The collection of flags $\bar{F}_i R_k$ defines a homogeneous, pointwise left bounded filtration $\bar{\chi}$. We claim that $\bar{\chi}$ is multiplicative. This follows since $\bar{F}_i R_k \subset R_k$ is the subspace spanned by $\lim_{\tau \rightarrow 0} \lambda(\tau) \cdot s$ for all $s \in F_i R_k$ and we have $\lambda(\tau)(s_1 s_2) = (\lambda(\tau)s_1)(\lambda(\tau)s_2)$. Moreover $\bar{\chi}$ is obviously preserved by the action of λ , hence by T .

To prove the statement about the weight functions note that we have $\dim \bar{F}_i R_k = \dim F_i R_k$ for all i and k . \square

As the Theorem shows the specialisation procedure preserves the class of polynomial filtrations. However, as explained in the Appendix, it does not preserve the class of finitely generated filtrations in general.

There is an alternative useful point of view on the filtration $\bar{\chi}$ of Definition 29. Let $\text{Rees}(\chi) \subset R$ be the Rees algebra of the finitely generated filtration χ corresponding to the test-configuration $(\mathcal{X}, \mathcal{L})$. A one parameter subgroup $\lambda: \mathbb{C}^* \rightarrow \text{Aut}(X, L)$ acts on R and on $R[t]$ (trivially on t) and we may define a $\mathbb{C}[t]$ -subalgebra $\text{Rees}^\lambda(\chi) \subset R$ by

$$\text{Rees}^\lambda(\chi) = \left\{ \lim_{\tau \rightarrow 0} \lambda(\tau) \cdot s : s \in \text{Rees}(\chi) \right\}.$$

Then $\bar{\chi}$ is precisely the filtration of R whose Rees algebra is $\text{Rees}^\lambda(\chi)$, i.e.

$$\bar{F}_i R_k = \{s \in R_k : t^i s \in \text{Rees}^\lambda(\chi)\}.$$

Let us now give the relevant definition for test-configurations.

Definition 31 (Specialisation of a test-configuration). *Let $(\mathcal{X}, \mathcal{L})$ be a test-configuration and T a torus in $\text{Aut}(X, L)$. Let χ be the finitely generated filtration corresponding to $(\mathcal{X}, \mathcal{L})$. A specialisation $(\mathcal{X}', \mathcal{L}')$ of $(\mathcal{X}, \mathcal{L})$ with respect to T is defined by first taking the specialisation $\bar{\chi}$ of χ in the sense of Definition 29, then by taking a finitely generated approximation of $\bar{\chi}$ as in Definition 23, choosing the exponent r so large that Proposition 24 applies.*

Theorem 32. *Let $(\mathcal{X}', \mathcal{L}')$ be a specialisation of $(\mathcal{X}, \mathcal{L})$ in the sense of Definition 31. Then $(\mathcal{X}', \mathcal{L}')$ is T -equivariant and we have*

$$\mathrm{DF}(\mathcal{X}', \mathcal{L}') = \mathrm{DF}(\mathcal{X}, \mathcal{L}), \quad \|(\mathcal{X}', \mathcal{L}')\|_{L^2} = \|(\mathcal{X}, \mathcal{L})\|_{L^2}.$$

Proof. This follows at once from Theorem 30. \square

Let us emphasise that the exponent of $(\mathcal{X}', \mathcal{L}')$ is in general different from the exponent of $(\mathcal{X}, \mathcal{L})$: even when $(\mathcal{X}', \mathcal{L}')$ is finitely generated, the specialisation does not take place in a fixed projective space depending only on $(\mathcal{X}, \mathcal{L})$.

4. TORUS EQUIVARIANT K-STABILITY

It is now easy to give a first result about K-semistability.

Proposition 33 (Equivariant K-semistability). *Fix a torus $T \subset \mathrm{Aut}(X, L)$ (not necessarily maximal). If (X, L) is K-semistable with respect to T -equivariant test-configurations then it is K-semistable.*

Proof. This is an immediate corollary of Theorem 32. \square

To obtain a result about K-stability we need to analyse more carefully the specialisation procedure. We will often discuss separately the case when $\bar{\chi}$ is finitely generated and when it is not. We will also need to assume that T is a maximal torus in $\mathrm{Aut}(X, L)$.

In the special situation when $\bar{\chi}$ is finitely generated we get a useful geometric consequence.

Lemma 34. *Suppose that $\mathrm{Rees}(\bar{\chi}) = \mathrm{Rees}^\lambda(\chi)$ is a finitely generated $\mathbb{C}[t]$ -subalgebra of $R[t]$. Then there exist an embedding $\iota: \mathcal{X} \rightarrow \mathbb{P}^N \times \mathbb{C}$ and a 1-parameter subgroup $\hat{\lambda}: \mathbb{C}^* \rightarrow \mathrm{GL}(N+1, \mathbb{C})$ such that*

- $\iota^* \mathcal{O}_{\mathbb{P}^N}(1) = \mathcal{L}^r$ for some $r \geq 1$,
- $\hat{\lambda}$ acting on \mathbb{P}^N preserves $\iota(\mathcal{X}_1) \cong X$ and restricts to the induced action of λ on it,
- the 1-parameter flat family of subschemes of $\mathbb{P}^N \times \mathbb{C}$ induced by $\hat{\lambda}$ (acting trivially on the second factor) has central fibre isomorphic to $\mathrm{Proj}(\mathrm{Rees}(\bar{\chi}))$ endowed with its natural Serre line bundle $\mathcal{O}(r)$.

In particular it follows that the central fibre $(\mathcal{X}'_0, \mathcal{L}'_0)$ is a flat 1-parameter degeneration of the central fibre $(\mathcal{X}_0, \mathcal{L}_0)$ (as closed subschemes of \mathbb{P}^N).

Proof. If $\mathrm{Rees}(\bar{\chi}) = \mathrm{Rees}^\lambda(\chi) \subset R[t]$ is a finitely generated $\mathbb{C}[t]$ -subalgebra there exists a finite set of elements σ_i of $\mathrm{Rees}(\chi)$ such that the limits $\lim_{\tau \rightarrow 0} \lambda(\tau) \cdot \sigma_i$ generate $\mathrm{Rees}(\bar{\chi})$. Since $\lambda(\tau)$ is $\mathbb{C}[t]$ -linear and we have $\lambda(\tau) \cdot (s_1 + s_2) = \lambda(\tau) \cdot s_1 + \lambda(\tau) \cdot s_2$ and $\lambda(\tau) \cdot (s_1 s_2) = (\lambda(\tau) \cdot s_1)(\lambda(\tau) \cdot s_2)$ for all $s_1, s_2 \in R$ we can choose our σ_i of the special form $\sigma_i = t^{p(i)} s_i$ where the s_i are homogeneous elements of R . Moreover we can assume that the elements $t^{p(i)} s_i$, $i = 0, \dots, N$ generate $\mathrm{Rees}(\chi)$. For a suitable

$r \geq 1$ the monomials \tilde{s}_j in our elements s_i of homogenous degree r generate the Veronese algebra $\tilde{R} = \bigoplus_{k \geq 0} R_{kr}$ (which is thus generated in degree 1) and so the corresponding elements $t^{p(j)} \tilde{s}_j$ generate the Veronese algebra $\bigoplus_{k \geq 0} (F_{kr} \tilde{R}) t^{kr}$ and their limits $t^{p(j)} \lim_{\tau \rightarrow 0} \lambda(\tau) \cdot \tilde{s}_j$ generate the Veronese algebra $\bigoplus_{k \geq 0} (\bar{F}_{kr} \tilde{R}) t^{kr}$.

With these assumptions we define a surjective morphism of $\mathbb{C}[t]$ -algebras

$$\phi: \mathbb{C}[\xi_0, \dots, \xi_N][t] \rightarrow \bigoplus_{k \geq 0} (F_{kr} \tilde{R}) t^{kr}$$

by $\phi(t) = t$, $\phi(\xi_i) = t^{p(i)} \tilde{s}_i$. Suppose that the action of λ is given by $\lambda(\tau) \cdot \tilde{s}_i = \sum_j a_{ij}(\tau) \tilde{s}_j$. We define a one parameter subgroup $\hat{\lambda}: \mathbb{C}^* \rightarrow GL(\mathbb{C}_1[\xi_0, \dots, \xi_N])$, acting on degree 1 elements by $\hat{\lambda}(\tau) \cdot \xi_i = \sum_j a_{ij}(\tau) \xi_j$, and extend its action trivially on t . The morphism ϕ induces the required embedding

$$\iota: \mathcal{X} = \text{Proj}_{\mathbb{C}[t]} \bigoplus_{k \geq 0} (F_{kr} \tilde{R}) t^{kr} \rightarrow \text{Proj}_{\mathbb{C}[t]} \mathbb{C}[\xi_0, \dots, \xi_N][t],$$

which intertwines the actions of λ and $\hat{\lambda}$. By construction the limit as $\tau \rightarrow 0$ of the flat family of closed subschemes of $\mathbb{P}^N \times \mathbb{C}$ given by

$$\hat{\lambda}(\tau) \cdot \iota \left(\text{Proj}_{\mathbb{C}[t]} \bigoplus_{k \geq 0} (F_{kr} \tilde{R}) t^{kr} \right)$$

is isomorphic to $\text{Proj}_{\mathbb{C}[t]} \bigoplus_{k \geq 0} (\bar{F}_{kr} \tilde{R}) t^{kr}$ and so it gives a copy of \mathcal{X}' embedded in $\mathbb{P}^N \times \mathbb{C}$ as a flat 1-parameter degeneration of \mathcal{X} .

To prove the statement on central fibres we look at the family of closed subschemes of \mathbb{P}^N given by

$$\hat{\lambda}(\tau) \cdot \iota(\mathcal{X}_0) = \hat{\lambda}(\tau) \cdot \iota \left(\text{Proj}_{\mathbb{C}[t]} \text{gr} \bigoplus_{k \geq 0} (F_{kr} \tilde{R}) t^{kr} \right).$$

Taking the flat closure of this 1-parameter family we obtain a closed subscheme $\mathcal{Y}_0 \subset \mathbb{P}^N$ whose underlying reduced subscheme $\mathcal{Y}_0^{\text{red}}$ is contained in $X'_0 \subset \mathbb{P}^N$. By flatness the Hilbert function of \mathcal{Y}_0 is the same as that of the central fibre $(\mathcal{X}_0, \mathcal{L}_0^r)$ and so the same as that of the general fibre (X, L^r) . Similarly the Hilbert function of $\mathcal{X}'_0 \subset \mathbb{P}^N$ is the same as that of $(\mathcal{X}'_0, \mathcal{L}'_0{}^r)$ and so the same as that of the general fibre (X, L^r) . As we have $\mathcal{Y}_0^{\text{red}} \subset \mathcal{X}'_0 \subset \mathbb{P}^N$ and $\mathcal{X}'_0, \mathcal{Y}_0 \subset \mathbb{P}^N$ have the same Hilbert functions we must actually have $\mathcal{Y}'_0 = \mathcal{X}'_0$ as required. \square

Proof of Theorem 2. Let $\bar{\chi}$ denote the T -invariant filtration constructed in Definition 29 (recall that T is now a maximal torus). We study two separate cases.

Case 1: the filtration $\bar{\chi}$ is not finitely generated. Consider the finitely generated approximations $\bar{\chi}^{(r)}$ of Definition 23. For sufficiently large r we have $\|\bar{\chi}^{(r)}\|_{L^2} = \|\bar{\chi}\|_{L^2}$ and $\text{DF}(\bar{\chi}^{(r)}) = \text{DF}(\bar{\chi})$.

The test-configuration $(\text{Proj}(\bar{\chi}^{(r)}), \mathcal{O}(1))$ is a product if and only if the filtration $\bar{\chi}^{(r)}$ on the homogeneous coordinate ring R coincides with the filtration induced by a holomorphic vector field in the Lie algebra of $\text{Aut}(X, L)$.

Suppose that for all r we have that $\bar{\chi}^{(r)}$ is induced by a holomorphic vector field ξ_r in the Lie algebra of $\text{Aut}(X, L)$. Since $\bar{\chi}^{(r)}$ is T -equivariant we know that ξ_r commutes with T . And since T is maximal ξ_r is contained in the Lie algebra of T . We claim that in fact ξ_r is a constant sequence $\xi_r = \xi$ for large r . To prove the claim, fix a value of k ; for large r all the ξ_r induce the same filtration on R_k because the filtrations $\bar{\chi}^{(r)}$ stabilise on R_k . Since all ξ_r lie in the same maximal torus and induce the same filtration, they must actually coincide. So for all large r the filtration $\bar{\chi}^{(r)}$ is induced by the same holomorphic vector field ξ , i.e. for all large r the sequence $\bar{\chi}^{(r)}$ is constant and equal to the filtration induced by ξ . But then $\bar{\chi}$ also equals the filtration induced by ξ and so is finitely generated, a contradiction.

Case 2: the filtration $\bar{\chi}$ is finitely generated. In this case Lemma 34 shows that for some r the central fibre $(\mathcal{X}'_0, \mathcal{L}'_0{}^r)$ is in the closure of the orbit of the central fibre $(\mathcal{X}_0, \mathcal{L}_0)$ in the relevant Hilbert scheme of subschemes of \mathbb{P}^N . On the other hand we are assuming that $(\mathcal{X}_0, \mathcal{L}_0^r)$ is not isomorphic to (X, L^r) so $(\mathcal{X}_0, \mathcal{L}_0^r)$ is contained in the closure of the orbit of (X, L^r) but not in the orbit itself. If $(\mathcal{X}'_0, \mathcal{L}'_0{}^r)$ is isomorphic to (X, L^r) it must lie in its orbit, and since it is a specialisation of $(\mathcal{X}_0, \mathcal{L}_0^r)$ the latter must also lie in the same orbit, a contradiction. \square

Proof of Theorem 3. As in the proof of Theorem 2 we specialise χ to the not necessarily finitely generated T -equivariant filtration $\bar{\chi}$ and discuss two separate cases.

Case 1: the filtration $\bar{\chi}$ is not finitely generated. Let $\bar{\chi}^{(r)}$ be the infinite sequence of T -equivariant, finitely generated approximations to $\bar{\chi}$ constructed in the proof of Theorem 2. These correspond to T -equivariant test-configurations $(\mathcal{X}^{(r)}, \mathcal{L}^{(r)})$. Let us denote by $(\hat{\mathcal{X}}^{(r)}, \hat{\mathcal{L}}^{(r)})$ their normalisations. These are still T -equivariant and a result of Ross and Thomas ([25] Remark 5.2) shows that

$$\text{DF}(\hat{\mathcal{X}}^{(r)}, \hat{\mathcal{L}}^{(r)}) \leq \text{DF}(\mathcal{X}^{(r)}, \mathcal{L}^{(r)}).$$

We claim that for sufficiently large r the test-configuration $(\hat{\mathcal{X}}^{(r)}, \hat{\mathcal{L}}^{(r)})$ is not a product.

The Rees algebra of $\bar{\chi}^{(r)}$ is a subalgebra $\text{Rees}(\bar{\chi}^{(r)}) \subset R[t]$ of the domain $R[t]$. Since X is normal by assumption $R[t]$ is integrally closed. So denoting by $\widehat{\text{Rees}(\bar{\chi}^{(r)})}$ the degree zero integral closure we have $\widehat{\text{Rees}(\bar{\chi}^{(r)})} \subset R[t]$. By the definition of normalisation in terms of an open affine covering ([15] exercise II.3.8), the standard local description of Proj ([15] Proposition II.2.5) and the fact that localisation with respect to a multiplicative system commutes with integral closure ([2] Proposition 5.12) we have the global description

$$(\hat{\mathcal{X}}^{(r)}, \hat{\mathcal{L}}^{(r)}) = \text{Proj}(\widehat{\text{Rees}(\bar{\chi}^{(r)})}, \mathcal{O}(1)).$$

Using Lemma 16 one checks that $\widehat{\text{Rees}(\bar{\chi}^{(r)})} \subset R[t]$ is in fact the Rees algebra of a filtration $\hat{\chi}^{(r)}$, so we find

$$(\hat{\mathcal{X}}^{(r)}, \hat{\mathcal{L}}^{(r)}) = \text{Proj}(\text{Rees}(\hat{\chi}^{(r)}), \mathcal{O}(1)).$$

Similarly let us denote by $\widehat{\text{Rees}(\bar{\chi})} \subset R[t]$ the degree zero integral closure of the Rees algebra of $\bar{\chi}$. By Lemma 16 this is in fact the Rees algebra of a filtration $\hat{\chi}$.

Since $\text{Rees}(\bar{\chi}^{(r)})$ is a subalgebra of $\text{Rees}(\bar{\chi})$, for all r there is a straightforward inclusion $\text{Rees}(\hat{\chi}^{(r)}) \subset \text{Rees}(\hat{\chi})$. Conversely if we fix $\sigma \in \text{Rees}(\hat{\chi}) = \widehat{\text{Rees}(\bar{\chi})}$ then σ is the root of a monic polynomial with coefficients in $\text{Rees}(\bar{\chi})$. Since the filtrations $\bar{\chi}^{(r)}$ converge pointwise to $\bar{\chi}$ these coefficients also lie in $\text{Rees}(\bar{\chi}^{(r)})$ for all sufficiently large r (depending on σ). It follows that σ is also contained in $\text{Rees}(\hat{\chi}^{(r)})$ for all sufficiently large r .

In particular we see that the filtrations induced by $\hat{\chi}^{(r)}$ on $H^0(X, L^k)$ stabilise to the filtration induced by $\hat{\chi}$ for all large r and fixed k . Assuming by contradiction that all the test-configurations $(\hat{\mathcal{X}}^{(r)}, \hat{\mathcal{L}}^{(r)})$ are products we can proceed as in the non-finitely generated case of the proof of Theorem 2. It follows that $\hat{\chi}$ is the filtration induced by a single holomorphic vector field and so is finitely generated.

However the Rees algebra $\text{Rees}(\hat{\chi})$ cannot be finitely generated. Recall that $\text{Rees}(\hat{\chi})$ is the degree zero integral closure $\widehat{\text{Rees}(\bar{\chi})}$ of $\text{Rees}(\bar{\chi})$. So there is an inclusion $\text{Rees}(\bar{\chi}) \subset \text{Rees}(\hat{\chi})$ which is an integral extension of $\mathbb{C}[t]$ -algebras. Integral extensions satisfy the going-up theorem for chains of prime ideals also in the non-noetherian case ([2] Theorem 5.11). Supposing $\text{Rees}(\hat{\chi})$ is a finitely generated $\mathbb{C}[t]$ -algebra then it is noetherian. But then the going-up theorem shows that $\text{Rees}(\bar{\chi})$ is noetherian ([2] exercise VII.12), a contradiction.

Case 2: the filtration $\bar{\chi}$ is finitely generated. Lemma 34 shows that the finitely generated filtration $\bar{\chi}$ corresponds to a T -equivariant test-configuration $(\mathcal{X}', \mathcal{L}')$ for (X, L) which is a 1-parameter flat degeneration of $(\mathcal{X}, \mathcal{L})$. Moreover the central fibre $(\mathcal{X}'_0, \mathcal{L}'_0)$ is a 1-parameter flat degeneration of $(\mathcal{X}_0, \mathcal{L}_0)$.

Let $(\hat{\mathcal{X}}', \hat{\mathcal{L}}')$ denote the normalisation of $(\mathcal{X}', \mathcal{L}')$. It is a T -equivariant test-configuration for (X, L) with normal total space. By [25] Remark 5.2 we have $\text{DF}(\hat{\mathcal{X}}', \hat{\mathcal{L}}') \leq \text{DF}(\mathcal{X}', \mathcal{L}')$. Thus it only remains to be seen that (assuming \mathcal{X} is normal and not a product) the central fibre $\hat{\mathcal{X}}'_0$ is not isomorphic to X , i.e. that $(\hat{\mathcal{X}}', \hat{\mathcal{L}}')$ is not a product test-configuration.

Recall that $X, \mathcal{X}_0, \mathcal{X}'_0$ all lie in the same projective space \mathbb{P}^N . If $V \subset \mathbb{P}^N$ is a subscheme we denote by $\eta(V)$ the Lie algebra of holomorphic vector fields on \mathbb{P}^N which are tangent to the smooth part of the reduced subscheme V^{red} .

If \mathcal{X} is normal and \mathcal{X}_0 is not isomorphic to X then \mathcal{X}_0 has an additional nontrivial holomorphic vector field given by the infinitesimal generator of

the \mathbb{C}^* -action on the test-configuration, that is

$$\dim \eta(\mathcal{X}_0) > \dim \eta(X).$$

Conversely if the normalisation $\hat{\mathcal{X}}'$ is a product then \mathcal{X}'_0 has no additional holomorphic vector fields, that is

$$\dim \eta(\mathcal{X}'_0) = \dim \eta(X).$$

But \mathcal{X}'_0 is a flat limit of \mathcal{X}_0 in \mathbb{P}^N , and the dimension of the space of holomorphic vector fields can only increase in such a family, i.e. we must have

$$\dim \eta(\mathcal{X}'_0) \geq \dim \eta(\mathcal{X}_0).$$

So $\hat{\mathcal{X}}'$ cannot be isomorphic to $X \times \mathbb{C}$. \square

5. L^2 PRODUCT AND NON-DEGENERACY CONDITION

Let χ be a filtration as in Definition 14. Then χ induces a filtration on the vector space $H^0(X, L^k)$ for all k , which we still denote by χ . Fixing k this induced filtration defines a parabolic subgroup $\mathcal{P}_k(\chi) \subset GL(H^0(X, L^k))$.

Lemma 35 ([22] Lemma 2.9). *Let χ_i , $i = 1, 2$ be filtrations. The intersection of parabolic subgroups $\mathcal{P}_k(\chi_1) \cap \mathcal{P}_k(\chi_2)$ contains some maximal torus \mathcal{T} . There exist unique 1-parameter subgroups α, β of \mathcal{T} such that their weight filtrations coincide with χ_1, χ_2 .*

Note that the maximal torus $\mathcal{T} \subset GL(H^0(X, L^k))$ is not in general unique. Let us denote by A_k, B_k the infinitesimal generators of the 1-parameter subgroups of Lemma 35.

Definition 36 (Product functions). *With the above notation we let*

$$P_{\chi_1, \chi_2}(k) = P(k) = \text{Tr}_{H^0(X, L^k)}(A_k B_k)$$

and

$$\bar{P}_{\chi_1, \chi_2}(k) = \bar{P}(k) = P(k) - \frac{w_{\chi_1}(k)w_{\chi_2}(k)}{h(k)}.$$

We define

$$\langle \chi_1, \chi_2 \rangle = \limsup_{k \rightarrow \infty} k^{-n-2} \bar{P}(k).$$

The definition does not depend on the choice of maximal torus \mathcal{T} since choosing a different maximal torus corresponds to conjugating both A_k, B_k by the same element of $GL(H^0(X, L^k))$.

Lemma 37. *If both χ_1 and χ_2 are polynomial filtrations then $\langle \chi_1, \chi_2 \rangle$ is finite.*

Proof. By the Cauchy-Schwarz inequality we have for every k

$$P_{\chi_1, \chi_2}(k) \leq (d_{\chi_1}(k)d_{\chi_2}(k))^{\frac{1}{2}}.$$

We divide by k^{n+2} and note that by the polynomiality assumption the right hand side is bounded. Taking the lim sup we obtain the Lemma. \square

In general we do not know if $P_{\chi_1, \chi_2}(k)$ is a polynomial for large k or if $\langle \chi_1, \chi_2 \rangle$ is finite.

The group $\text{Aut}(X, L)$ acts naturally on the set of all filtrations. Explicitly if $g \in \text{Aut}(X, L)$ we define $(gF_i)H^0(X, L^k) = g(F_i H^0(X, L^k))$ for every k and i ; it is straightforward to check that the new filtration satisfies Definition 14. The following elementary observation turns out to be quite useful.

Lemma 38. *The scalar product of Definition 36 is invariant under the natural action of $\text{Aut}(X, L)$ on the set of all filtrations.*

Definition 36 applies in particular to finitely generated filtrations and so to test-configurations. In this case we write the scalar product as

$$\langle (\mathcal{X}, \mathcal{L}), (\mathcal{X}', \mathcal{L}') \rangle.$$

If $(\mathcal{X}', \mathcal{L}')$ in the product test-configuration induced by a 1-parameter subgroup β of $\text{Aut}(X, L)$ we write $\langle (\mathcal{X}, \mathcal{L}), \beta \rangle$.

Lemma 39. *Let $\beta : \mathbb{C}^* \rightarrow \text{Aut}(X, L)$ be a 1-parameter subgroup and let $(\mathcal{X}, \mathcal{L})$ be a β -equivariant test-configuration. Denote by $\alpha, \tilde{\beta}$ the corresponding \mathbb{C}^* -actions on the central fibre and by $\langle \alpha, \tilde{\beta} \rangle$ their L^2 scalar product in the sense of [31]. Then we have*

$$\langle (\mathcal{X}, \mathcal{L}), \beta \rangle = \langle \alpha, \tilde{\beta} \rangle.$$

Proof. The test-configuration $(\mathcal{X}, \mathcal{L})$ corresponds to a finitely generated filtration χ . In this case β gives a natural \mathbb{C}^* -action on $H^0(X, L^k)$ for all k , and we can find a second \mathbb{C}^* -action α'_k on $H^0(X, L^k)$ which commutes with β and whose weight filtration coincides with the filtration induced by χ , which we denote by $F_i H^0(X, L^k)$. Let us denote by A_k, B_k the infinitesimal generators of α'_k , respectively β . Now α'_k, β preserve all filtered pieces $F_i H^0(X, L^k)$ and so induce naturally \mathbb{C}^* -actions on $F_i(H^0(X, L^k))/F_{i-1}(H^0(X, L^k))$ for all i , whose infinitesimal generators are induced by A_k, B_k . We have

$$\begin{aligned} \text{Tr}(A_k B_k) &= \sum_i \text{Tr}(A_k B_k|_{F_i H^0(X, L^k)}) - \text{Tr}(A_k B_k|_{F_{i-1} H^0(X, L^k)}) \\ &= \sum_i \text{Tr}(A_k B_k|_{F_i(H^0(X, L^k))/F_{i-1}(H^0(X, L^k))}). \end{aligned} \quad (5.1)$$

The central fibre $(\mathcal{X}_0, \mathcal{L}_0)$ is given by

$$(\text{Proj}(\bigoplus_{i,k} F_i(H^0(X, L^k))/F_{i-1}(H^0(X, L^k))), \mathcal{O}(1)),$$

with the actions of weight $-i$ of $\alpha_k, \tilde{\beta}$ corresponding to the grading by i , respectively to the induced action of β on $F_i(H^0(X, L^k))/F_{i-1}(H^0(X, L^k))$. So we have

$$H^0((\mathcal{X}_0, \mathcal{L}_0^k)) = \bigoplus_i F_i(H^0(X, L^k))/F_{i-1}(H^0(X, L^k))$$

and the total weight of the product of infinitesimal generators on $H^0((\mathcal{X}_0, \mathcal{L}_0^k))$ can be computed as

$$\sum_i \text{Tr } A_k B_k|_{F_i(H^0(X, L^k))/F_{i-1}(H^0(X, L^k))}.$$

The result follows by comparing with (5.1). \square

Picking a basis β_i of \mathbb{C}^* -actions generating T which is orthogonal with respect to the scalar product $\langle -, - \rangle$ we can define the L^2 projection of $(\mathcal{X}, \mathcal{L})$ along T as

$$(\mathcal{X}, \mathcal{L})_T = \sum_i \frac{\langle (\mathcal{X}, \mathcal{L}), \beta_i \rangle}{\|\beta_i\|^2} \beta_i.$$

Note that we are abusing notation slightly: the right hand side is a one parameter subgroup of $\text{Aut}(X, L)$ (up to a rational multiple) while the test-configuration on the left hand side is the corresponding product test-configuration. (It is not obvious that the right hand side is indeed such a rational multiple, but this will be shown in the proof of Theorem 9). This definition continues to make sense also if we replace $(\mathcal{X}, \mathcal{L})$ with a polynomial non-finitely generated filtration; in this case, $(\mathcal{X}, \mathcal{L})_T$ is still a test-configuration, i.e. a finitely generated filtration. We have

$$\|(\mathcal{X}, \mathcal{L})_T\|_{L^2}^2 = \sum_i \frac{\langle (\mathcal{X}, \mathcal{L}), \beta_i \rangle^2}{\|\beta_i\|^2}.$$

The projection does not depend on the choice of basis. By Lemma 39 when $(\mathcal{X}, \mathcal{L})$ is \mathbb{C}^* -equivariant, with a \mathbb{C}^* -action α commuting with an action of T generated by $\tilde{\beta}_i$ we have

$$(\mathcal{X}, \mathcal{L})_T = \sum_i \frac{\langle \alpha, \tilde{\beta}_i \rangle}{\|\tilde{\beta}_i\|^2} \tilde{\beta}_i. \quad (5.2)$$

Since α and the $\tilde{\beta}_i$ commute, in this case it also makes sense to define the L^2 orthogonal to T , namely we let $(\mathcal{X}, \mathcal{L})_T^\perp$ be the test-configuration whose total space is $(\mathcal{X}, \mathcal{L})$ endowed with the \mathbb{C}^* -action

$$\alpha - \sum_i \frac{\langle \alpha, \tilde{\beta}_i \rangle}{\|\tilde{\beta}_i\|^2} \tilde{\beta}_i. \quad (5.3)$$

The previous expression is a well-defined \mathbb{C}^* -action on $(\mathcal{X}, \mathcal{L})$ because $(\mathcal{X}, \mathcal{L})$ is T equivariant; without the T -equivariance such a definition would not make sense. Note that $(\mathcal{X}, \mathcal{L})_T^\perp$ is not, in general, a product test-configuration. By Lemma 39 $(\mathcal{X}, \mathcal{L})_T^\perp$ coincides with the L^2 orthogonal of α to T as introduced in [31]. We now state our non-degeneracy condition.

Definition 40 (L^2 non-degeneracy condition). *We say that a test-configuration $(\mathcal{X}, \mathcal{L})$ for the polarised variety (X, L) is non-degenerate with respect to a torus $T \subset \text{Aut}(X, L)$ if we have $\|(\mathcal{X}, \mathcal{L})_T\|_{L^2} < \|(\mathcal{X}, \mathcal{L})\|_{L^2}$. Thus*

a T -equivariant test-configuration $(\mathcal{X}, \mathcal{L})$ is non-degenerate with respect to T if and only if $\|(\mathcal{X}, \mathcal{L})_T^\perp\|_{L^2} > 0$.

We give here a partial characterisation of this non-degeneracy condition. A complete description for normal test-configurations is provided by Proposition 44.

Lemma 41. *Let $(\mathcal{X}, \mathcal{L})$ be a T -equivariant test-configuration. Then $(\mathcal{X}, \mathcal{L})$ is degenerate with respect to T if and only if its normalisation $(\hat{\mathcal{X}}, \hat{\mathcal{L}})$ is induced by a holomorphic vector field lying in T .*

Proof. A T -equivariant test-configuration $(\mathcal{X}, \mathcal{L})$ is degenerate with respect to T if and only if $\|(\mathcal{X}, \mathcal{L})_T^\perp\|_{L^2} = 0$. Note that $(\mathcal{X}, \mathcal{L})_T^\perp$ is still a test-configuration for (X, L) , with the same underlying total space \mathcal{X} but different \mathbb{C}^* -action given by (5.3). Then according to [12] Theorem 4.7 and [5] Theorem A (iii) the condition $\|(\mathcal{X}, \mathcal{L})_T^\perp\|_{L^2} = 0$ holds if and only if the normalisation of the test-configuration $(\mathcal{X}, \mathcal{L})_T^\perp$ is \mathbb{C}^* -equivariantly isomorphic to the product $X \times \mathbb{C}$ with trivial \mathbb{C}^* -action. This means precisely that the pullback of the vector field (5.3) to the normalisation $(\hat{\mathcal{X}}, \hat{\mathcal{L}})$ of $(\mathcal{X}, \mathcal{L})$ vanishes identically, or equivalently that the normalisation $(\hat{\mathcal{X}}, \hat{\mathcal{L}})$ is the product test-configuration $X \times \mathbb{C}$ endowed with the possibly nontrivial \mathbb{C}^* -action (5.2), with β_i given by the trivial extension of β_i to $X \times \mathbb{C}$. \square

Let us describe how this degeneracy condition behaves with respect to the specialisation procedure defined in Section 3.

Lemma 42. *A test-configuration is degenerate with respect to a maximal torus T of $\text{Aut}(X, L)$ if and only if its specialisation with respect to T in the sense of Definition 31 is degenerate with respect to T . The same statement holds for filtrations.*

Proof. Let us first prove the claim for arbitrary filtrations. It is enough to show that

$$\langle \chi, \gamma \rangle = \langle \chi_\lambda, \gamma \rangle$$

where γ is a one parameter subgroup of T and λ is the one parameter subgroup that we used in Definition 29. But this follows at once because the pairing is $\text{Aut}(X, L)$ -invariant and γ commutes with λ .

To prove the statement about test-configurations it remains to consider the case when χ_λ is not finitely generated. By Lemma 41 it is enough to show that if $\bar{\chi}$ is not finitely generated then the normalisation of its finitely generated approximations are not all product test-configurations. This is shown in the proof of Theorem 3. \square

We can now show that the L^2 non-degeneracy condition does not depend on the choice of T in $\text{Aut}(X, L)$.

Proposition 43. *Let $(\mathcal{X}, \mathcal{L})$ be a test-configuration which is degenerate with respect to a maximal torus T of $\text{Aut}(X, L)$. Then it is degenerate with respect to any other maximal torus H of $\text{Aut}(X, L)$.*

Proof. Let g be an element of $\text{Aut}(X, L)$ such that $gT = H$, where the action is the conjugation. Pick an orthogonal basis β_i of one parameter subgroups of T and let $\gamma_i = g\beta_i$ be the corresponding basis for H . We are going to show that $\langle (\mathcal{X}, \mathcal{L}), \beta_i \rangle = \langle (\mathcal{X}, \mathcal{L}), \gamma_i \rangle$ for every i . This is enough because β_i and γ_i have the same norm.

Pick a generic one parameter subgroup λ of T and let $(\mathcal{X}, \mathcal{L})_\lambda$ be the specialisation of $(\mathcal{X}, \mathcal{L})$ via λ , as in Definition 31; remark that $g\lambda$ is a generic one parameter subgroup of H . We have

$$\langle (\mathcal{X}, \mathcal{L}), \beta_i \rangle = \langle (\mathcal{X}, \mathcal{L})_\lambda, \beta_i \rangle = \langle (\mathcal{X}, \mathcal{L})_{g\lambda}, \gamma_i \rangle = \langle (\mathcal{X}, \mathcal{L}), \gamma_i \rangle$$

where we used the fact that the pairing is $\text{Aut}(X, L)$ -invariant. \square

The proofs of our main results involving the L^2 nondegeneracy condition are now quite straightforward.

Proof of Theorem 9. Let $(\mathcal{X}', \mathcal{L}')$ be a specialisation of $(\mathcal{X}, \mathcal{L})$ in the sense of Definition 31. By Theorem 32 it only remains to show that $\|(\mathcal{X}', \mathcal{L}')_T\|_{L^2} = \|(\mathcal{X}, \mathcal{L})_T\|_{L^2}$. Let χ be the finitely generated filtration corresponding to $(\mathcal{X}, \mathcal{L})$ and let $\bar{\chi}$ be its specialisation under the action of T in the sense of Definition 29. We have already observed in the proof of Lemma 42 that we have $\langle \chi, \gamma \rangle = \langle \bar{\chi}, \gamma \rangle$ for all one parameter subgroups γ of T . In particular we have $\|\bar{\chi}_T\|_{L^2} = \|\chi_T\|_{L^2}$. The theorem follows if we can prove that $\|\bar{\chi}_T^{(r)}\|_{L^2} = \|\bar{\chi}_T\|_{L^2}$ for sufficiently large r , where $\bar{\chi}^{(r)}$ denotes a finitely generated approximation as in Definition 23. To prove the claim recall that $\bar{\chi}^{(r)}$ induces the same filtration on R_k as $\bar{\chi}$ for all $k \leq r$. It follows that if γ is a one parameter subgroup of T , by choosing r large enough we can make sure that $P_{\bar{\chi}, \gamma}(k) = P_{\bar{\chi}^{(r)}, \gamma}(k)$ for arbitrarily many values of k . On the other hand $\bar{\chi}^{(r)}$ corresponds to a T -equivariant test-configuration and so $P_{\bar{\chi}^{(r)}, \gamma}(k)$ is a polynomial in k of degree $n+2$ for all large k . Thus $P_{\bar{\chi}, \gamma}(k)$ must also be a polynomial in k of degree $n+2$ for all large k , and for sufficiently large r we have $P_{\bar{\chi}, \gamma}(k) = P_{\bar{\chi}^{(r)}, \gamma}(k)$ for all large k . This gives at once $\langle \bar{\chi}, \gamma \rangle = \langle \bar{\chi}^{(r)}, \gamma \rangle$ for all γ in T and so in particular $\|\bar{\chi}_T^{(r)}\|_{L^2} = \|\bar{\chi}_T\|_{L^2}$. \square

Proof of Theorem 12. This follows from the same argument given in the proof of Theorem 9. \square

Finally we show the equivalence of the L^2 non-degeneracy condition with that of being a product test-configuration, when the total space is normal.

Proposition 44. *A normal test-configuration $(\mathcal{X}, \mathcal{L})$ is degenerate with respect to a maximal torus T in $\text{Aut}(X, L)$ if and only if the central fibre \mathcal{X}_0 is isomorphic to X .*

Proof. Let $(\mathcal{X}', \mathcal{L}')$ be the specialisation of $(\mathcal{X}, \mathcal{L})$ to a T -equivariant test-configuration as in Definition 31. By Lemma 42 the test-configuration $(\mathcal{X}', \mathcal{L}')$ is degenerate if and only if $(\mathcal{X}, \mathcal{L})$ is degenerate. According to

Lemma 41 $(\mathcal{X}', \mathcal{L}')$ is degenerate if and only if the central fibre of its normalisation is isomorphic to X . On the other hand it is shown in the proof of Theorem 3 that \mathcal{X}_0 is isomorphic to X if and only if the central fibre of the normalisation of $(\mathcal{X}', \mathcal{L}')$ is isomorphic to X . \square

6. THE PSEUDO-METRIC ON THE SPACE OF TEST-CONFIGURATIONS

In this section we briefly recall how the set of (equivalence classes of) test-configurations can be seen as a limit of spherical buildings (a point of view due to Odaka) and we prove Proposition 11 relating our scalar product to a natural pseudo-metric on this space.

Definition 45 (Weighted flag). *Let V be a vector space. A weighted flag is the data of a flag*

$$\{0\} \subset F_1V \subset F_2V \cdots F_{\ell-1}V \subset F_{\ell}V = V$$

and real weights $w = (w_1, \dots, w_{\ell})$ such that $w_i < w_{i+1}$. We identify two weight systems w_i, w'_i on the same flag if either $w_i = cw'_i$ or $w_i - w'_i = c$ for some constant c . We say that a flag is trivial if its length ℓ is equal to 1.

Definition 46 (Spherical building/Flag complex). *The spherical building Δ is the space of non-trivial weighted flags of V . Its rational points $\Delta(\mathbb{Q})$ correspond to flags with rational weights.*

Every one parameter subgroup of $GL(V)$ induces a rational weighted flag. Clearly points of $\Delta(\mathbb{Q})$ represent equivalence classes of non-central one parameter subgroups $\lambda: \mathbb{C}^* \rightarrow GL(V)$ modulo the equivalence relations

- for every pair of integers n and m we have the relation $\lambda^n \sim \lambda^m$;
- for every central 1PS γ we have $\lambda \sim \lambda\gamma$;
- for every g in $\mathcal{P}(\lambda)$, we have $\lambda \sim g\lambda g^{-1}$, where $\mathcal{P}(\lambda)$ is the parabolic subgroup of $GL(V)$ of elements preserving the flag induced by λ .

When $V = H^0(X, L^r)^{\vee}$ it is well-known that a one parameter subgroup of $GL(V)$ induces an exponent r test-configuration and, conversely, any exponent r test-configuration arises in this way. The precise link at the level of spherical buildings is given by the following result of Odaka.

Theorem 47 (Theorem 2.3 [23]). *Let $V = H^0(X, L^r)^{\vee}$. Points of $\Delta(\mathbb{Q})$ are in a bijective correspondence with exponent r non-trivial test-configurations of (X, L) , modulo base changes $z \mapsto z^p$ and change of the linearisation.*

As nicely explained in [26], the spherical building has a structure of simplicial complex. The vertexes of this complex are proper vector subspaces of V . A set of vertexes forms a face if the corresponding vector subspaces form a flag. In this way each simplex represents a non-weighted flag; the coordinates on the simplex are the weights. This complex is not even locally finite, hence it is far from being compact. However it is covered by *apartments*. These correspond to maximal tori of $GL(V)$; an apartment may be defined as the closure of the set of one parameter subgroups of a

maximal torus. Apartments turn out to be simplicial spheres of dimension $\dim V - 2$. Essentially because of Lemma 35 one can often reduce the proof of a statement about Δ to the proof of a statement on a single apartment.

As explained in [22] the flag complex also comes with a natural metric structure. Apartments are geodesically complete with respect to this metric. Let us sketch the construction of an apartment A as a metric space. Pick a maximal torus T and let M be the space of one parameter subgroups of T with determinant one, tensored by \mathbb{R} . The Killing form induces a natural norm on M . The apartment A is the unitary sphere in M endowed with the induced metric. The equivalence class of a one parameter subgroup in A is given by the radial projection from $M \setminus \{0\}$ to A . Because of Lemma 35, a pair of points is always contained in an apartment; hence the geodesic between two points is contained in an apartment. In particular we have $\text{diam}(\Delta) \leq \pi$.

Proof of Proposition 11. Suppose that $[\lambda], [\mu]$ are points in $|\Delta(H^0(X, L^r)^\vee)|_{\mathbb{Q}}$ corresponding to (possibly tensor powers of) two test-configurations $(\mathcal{X}, \mathcal{L})$, $(\mathcal{X}', \mathcal{L}')$. According to [22] (page 59) the distance function ρ_{rk} is defined by picking commuting, special linear representatives λ', μ' for the classes $[\lambda^{\otimes k}]$, $[\mu^{\otimes k}]$ and setting

$$\rho_{rk}((\mathcal{X}, \mathcal{L}), (\mathcal{X}', \mathcal{L}')) = \arccos \frac{1}{2} \left(\frac{\|\lambda' + \mu'\|^2}{\|\lambda'\| \|\mu'\|} - \frac{\|\lambda'\|}{\|\mu'\|} - \frac{\|\mu'\|}{\|\lambda'\|} \right)$$

where $\|\cdot\|$ denotes the standard Killing norm on 1-parameter subgroups of $SL(H^0(X, L^{rk})^\vee)$. This means that if A'_{rk}, B'_{rk} are infinitesimal generators of λ', μ' we have

$$\begin{aligned} \rho_{rk}((\mathcal{X}, \mathcal{L}), (\mathcal{X}', \mathcal{L}')) &= \arccos \frac{1}{2} \left(\frac{\text{Tr}(A'_{rk} + B'_{rk})^2}{(\text{Tr}(A'^2_{rk}))^{\frac{1}{2}} (\text{Tr}(B'^2_{rk}))^{\frac{1}{2}}} \right. \\ &\quad \left. - \frac{(\text{Tr}(A'^2_{rk}))^{\frac{1}{2}}}{(\text{Tr}(B'^2_{rk}))^{\frac{1}{2}}} - \frac{(\text{Tr}(B'^2_{rk}))^{\frac{1}{2}}}{(\text{Tr}(A'^2_{rk}))^{\frac{1}{2}}} \right) \\ &= \arccos \frac{\text{Tr}(A'_{rk} B'_{rk})}{(\text{Tr}(A'^2_{rk}))^{\frac{1}{2}} (\text{Tr}(B'^2_{rk}))^{\frac{1}{2}}}. \end{aligned}$$

Note that λ', μ' induce 1-parameter subgroups acting on the dual space $H^0(X, L^k)$. Their dual weight filtrations on $H^0(X, L^k)$ can be extended to filtrations χ, χ' of R which, after shifting the weights, induce the test-configurations $(\mathcal{X}, \mathcal{L})$, respectively $(\mathcal{X}', \mathcal{L}')$. So denoting by A_k, B_k the infinitesimal generators of these dual, shifted 1-parameter subgroups (no longer special linear) we have

$$P_{\chi, \chi'}(k) = \text{Tr} \left(\left(A_k - \frac{w_\chi(k)}{h(k)} \right) \left(B_k - \frac{w_{\chi'}(k)}{h(k)} \right) \right),$$

and we see that

$$P_{\chi, \chi'}(k) = \text{Tr}(A'_k B'_k).$$

Similarly we have

$$\text{Tr}(A_k'^2) = d_\chi(k)$$

where $d_\chi(k)$ is the trace squared function. The upshot is that we have

$$\rho_{rk}((\mathcal{X}, \mathcal{L}), (\mathcal{X}', \mathcal{L}')) = \arccos \frac{P_{\mathcal{X}, \mathcal{X}'}(k)}{\sqrt{d_\chi(k)d_{\chi'}(k)}}.$$

We can now compute the limit

$$\begin{aligned} \rho((\mathcal{X}, \mathcal{L}), (\mathcal{X}', \mathcal{L}')) &= \liminf_{k \rightarrow \infty} \rho_{rk}((\mathcal{X}, \mathcal{L}), (\mathcal{X}', \mathcal{L}')) \\ &= \arccos \limsup_{k \rightarrow \infty} \frac{P_{\mathcal{X}, \mathcal{X}'}(k)}{\sqrt{d_\chi(k)d_{\chi'}(k)}} \\ &= \arccos \frac{\langle (\mathcal{X}, \mathcal{L}), (\mathcal{X}', \mathcal{L}') \rangle}{\|(\mathcal{X}, \mathcal{L})\|_{L^2} \|(\mathcal{X}', \mathcal{L}')\|_{L^2}}. \end{aligned} \quad (6.1)$$

In the last equality we used that $\|(\mathcal{X}, \mathcal{L})\|_{L^2}$ and $\|(\mathcal{X}', \mathcal{L}')\|_{L^2}$ do not vanish. \square

Remark. Note that if one of the two norms vanishes in (6.1) we still have

$$\rho((\mathcal{X}, \mathcal{L}), (\mathcal{X}', \mathcal{L}')) = \arccos \limsup_{k \rightarrow \infty} \frac{P_{\mathcal{X}, \mathcal{X}'}(k)}{\sqrt{d_\chi(k)d_{\chi'}(k)}},$$

but we cannot compute this using only $\langle (\mathcal{X}, \mathcal{L}), (\mathcal{X}', \mathcal{L}') \rangle$ and the norms.

7. APPLICATIONS TO CSCK POLARISED MANIFOLDS

In this section we prove our results about the K-stability of constant scalar curvature polarised manifolds, Theorems 8 and 10. The theorems have almost identical proofs.

Proof of Theorem 10. Suppose that (X, L) is a constant scalar curvature polarised manifold. Pick a maximal torus $T \subset \text{Aut}(X, L)$ and let $(\mathcal{X}, \mathcal{L})$ be a test-configuration for (X, L) with $\|(\mathcal{X}, \mathcal{L})_T\|_{L^2} < \|(\mathcal{X}, \mathcal{L})\|_{L^2}$.

Since (X, L) is K-semistable [14] it is enough to show that we cannot have $\text{DF}(\mathcal{X}, \mathcal{L}) = 0$. We argue by contradiction assuming $\text{DF}(\mathcal{X}, \mathcal{L}) = 0$.

According to Theorem 9 we can find a T -equivariant test-configuration $(\mathcal{X}', \mathcal{L}')$ for (X, L) with $\text{DF}(\mathcal{X}', \mathcal{L}') = 0$ and $\|(\mathcal{X}', \mathcal{L}')^\perp_T\|_{L^2} > 0$. Denote the corresponding \mathbb{C}^* -action on $(\mathcal{X}', \mathcal{L}')$ by α . As $(\mathcal{X}', \mathcal{L}')$ is T -equivariant there are \mathbb{C}^* -actions $\tilde{\beta}_i$ on $(\mathcal{X}', \mathcal{L}')$, preserving the fibres, commuting with each other and with α , and extending the action of an orthogonal basis of 1-parameter subgroups β_i of $\text{Aut}(X, L)$. Fixing i , the total space $(\mathcal{X}', \mathcal{L}')$ endowed with the \mathbb{C}^* -action $\alpha \pm \tilde{\beta}_i$ is a test-configuration for (X, L) , with Donaldson-Futaki invariant

$$\begin{aligned} \text{DF}(\alpha \pm \tilde{\beta}_i) &= \text{DF}(\alpha) \pm \text{DF}(\tilde{\beta}_i) \\ &= \pm \text{DF}(\tilde{\beta}_i) \end{aligned}$$

(the first equality follows since $\alpha, \tilde{\beta}_i$ are commuting \mathbb{C}^* -actions on the same polarised scheme). Since we are assuming that (X, L) is constant scalar curvature we know it is K-semistable and so we must have $\text{DF}(\tilde{\beta}_i) = 0$ for all i . Recalling that $(\mathcal{X}', \mathcal{L}')_{\tilde{T}}^\perp$ is the test-configuration with total space $(\mathcal{X}', \mathcal{L}')$ endowed with \mathbb{C}^* -action

$$\alpha - \sum_i \frac{\langle \alpha, \tilde{\beta}_i \rangle}{\|\tilde{\beta}_i\|^2} \tilde{\beta}_i$$

we see that $\text{DF}(\mathcal{X}', \mathcal{L}')_{\tilde{T}}^\perp = 0$.

By Lemma 41 the normalisation $(\hat{\mathcal{X}}', \hat{\mathcal{L}}')_{\tilde{T}}^\perp$ of $(\mathcal{X}', \mathcal{L}')_{\tilde{T}}^\perp$ is not the product $X \times \mathbb{C}$ endowed with the trivial action. Equivalently $(\hat{\mathcal{X}}', \hat{\mathcal{L}}')_{\tilde{T}}^\perp$ is not isomorphic to $X \times \mathbb{C}$ (with trivial action) in codimension 1, i.e. outside a closed subscheme $Z \subset \hat{\mathcal{X}}'$ with $\text{codim}(Z) \geq 2$. Then by [29] section 3 there exists a point $p \in (\mathcal{X}'_1, \mathcal{L}'_1)$ which is fixed by the maximal torus T , and such that denoting by $\overline{\alpha \cdot p}$ the closure of the orbit of p in $(\mathcal{X}', \mathcal{L}')$ we have

$$\begin{aligned} \text{DF}(\text{Bl}_{\overline{\alpha \cdot p}} \mathcal{X}', \mathcal{L}' - \epsilon \mathcal{E})_{\tilde{T}}^\perp &= \text{DF}(\mathcal{X}', \mathcal{L}')_{\tilde{T}}^\perp - C\epsilon^{n-1} + O(\epsilon^n) \\ &= -C\epsilon^{n-1} + O(\epsilon^n) \end{aligned} \quad (7.1)$$

for some constant $C > 0$. Here $(\text{Bl}_{\overline{\alpha \cdot p}} \mathcal{X}', \mathcal{L}' - \epsilon \mathcal{E})$ is the test-configuration for $(\text{Bl}_p X, L - \epsilon E)$ (E, \mathcal{E} denoting the exceptional divisors) induced by blowing up the orbit $\overline{\alpha \cdot p}$ in \mathcal{X}' with sufficiently small rational parameter $\epsilon > 0$. Since p is fixed by T there is a natural inclusion $T \subset \text{Aut}(\text{Bl}_p X, L - \epsilon E)$ and then $(\text{Bl}_{\overline{\alpha \cdot p}} \mathcal{X}', \mathcal{L}' - \epsilon \mathcal{E})_{\tilde{T}}^\perp$ denotes the L^2 orthogonal to T in the usual sense.

As explained in [29] Theorem 2.4 a well-known result of Arezzo, Pacard and Singer [1] implies that the polarised manifold $(\text{Bl}_p X, L - \epsilon E)$ admits an extremal metric in the sense of Calabi. The semistability result of [31] shows that we must have $\text{DF}(\text{Bl}_{\overline{\alpha \cdot p}} \mathcal{X}', \mathcal{L}' - \epsilon \mathcal{E})_{\tilde{T}}^\perp \geq 0$. But this contradicts (7.1), so we must have in fact $\text{DF}(\mathcal{X}, \mathcal{L}) > 0$ as claimed. \square

Proof of Theorem 8. The same proof as for Theorem 10 works under the alternative assumption that $(\mathcal{X}, \mathcal{L})$ has normal total space and $(\mathcal{X}_0, \mathcal{L}_0)$ is not isomorphic to (X, L) . Indeed in this case we can assume by Theorem 3 that $(\mathcal{X}', \mathcal{L}')$ has normal total space and is not induced by a holomorphic vector field, so the same must hold for $(\mathcal{X}', \mathcal{L}')_{\tilde{T}}^\perp$. The rest of the argument is unchanged. \square

Remark. The proof of the main result of [29] (Theorem 1.4) shows that if (X, L) is extremal and $T \subset \text{Aut}(X, L)$ is a maximal torus then we have $\text{DF}(\mathcal{X}, \mathcal{L})_{\tilde{T}}^\perp > 0$ for all T -equivariant test-configurations whose normalisation is not induced by a holomorphic vector field in T (or equivalently, which are not isomorphic to such a product outside a closed subscheme of codimension at least 2). If the assumption is dropped there are counterexamples. Note that Theorem 1.4 in [29] is mistakenly stated without this assumption. See [19] Remark 4 and the note [28] for further discussion.

APPENDIX

In this appendix we present an example of a test-configuration $(\mathcal{X}, \mathcal{L})$ with a 1-parameter subgroup $\lambda: \mathbb{C}^* \rightarrow \text{Aut}(X, L)$ such that the λ -equivariant filtration $\bar{\chi}$ constructed in the proof of Proposition 33 is not finitely generated. This is done by adapting a well-known example in the literature on canonical bases of subalgebras, due to Robbiano and Sweedler ([24] Example 1.20).

Consider the polynomial algebra $\mathbb{C}[t][x, y]$ over the ring $\mathbb{C}[t]$ and let A denote the $\mathbb{C}[t]$ -subalgebra generated by

$$t(x + y), txy, txy^2, t^2y.$$

Then $A \subset R[t]$ is the Rees algebra of a homogeneous, multiplicative, pointwise left bounded finitely generated filtration χ of the homogeneous coordinate ring $R = \mathbb{C}[x, y]$ of the projective line $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$. So $\text{Proj}_{\mathbb{C}[t]} A$ endowed with its natural Serre bundle $\mathcal{O}(1)$ is a test-configuration for \mathbb{P}^1 . Consider the 1-parameter subgroup $\lambda: \mathbb{C}^* \rightarrow SL(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)))$ acting by

$$\lambda(\tau) \cdot x = \tau^{-1}x, \quad \lambda(\tau) \cdot y = \tau y.$$

We let $\bar{\chi}$ be the limit of χ under the action of λ as in the proof of Proposition 33.

Proposition 48. *The limit filtration $\bar{\chi}$ is not finitely generated.*

Proof. The 1-parameter subgroup λ induces a term ordering $>$ on the $\mathbb{C}[t]$ -algebra $\mathbb{C}[t][x, y]$ which is compatible with the graded $\mathbb{C}[t]$ -algebra structure and for which we have $x > y$. Let us denote the initial term of an element $\sigma \in \mathbb{C}[t][x, y]$ by $\text{in}_> \sigma$. The Rees algebra $\text{Rees}(\bar{\chi})$ coincides with the initial algebra of A defined by

$$\text{in}_> A = \{\text{in}_> \sigma : \sigma \in A\}.$$

We show that $\text{in}_> A$ is not finitely generated. The proof follows closely the original argument in [24] Example 1.20.

Claim 1. *The algebra A contains all the monomials of the form $t^{n-1}xy^n$ for $n \geq 3$, and does not contain elements which have a homogeneous component of the form t^kxy^n for $k < n-1$. In particular no element of A can have initial term of the form t^kxy^n for $k < n-1$. To check the first statement we observe that we have for $n \geq 3$*

$$t^{n-1}xy^n = t(x + y)t^{n-2}xy^{n-1} - t(xy)t^{n-3}xy^{n-2}$$

and then argue by induction starting from the fact that A contains the monomials $t(x + y), txy, txy^2$. For the second statement it is enough to check that A does not contain t^kxy^n for $k < n-1$ (since A is a graded subalgebra). This is a simple check.

Claim 2. *The algebra A does not contain elements which have a homogeneous component of the form t^ky^j for $k \leq j$. In particular no element of A can have initial term of the form t^ky^j for $k \leq j$. Since A is a graded subalgebra it is enough to show that t^ky^j cannot belong to A if $k \leq j$. All the*

elements of A are of the form $f(t(x+y), txy, txy^2, t^2y)$ where $f(x_1, x_2, x_3, x_4)$ is a polynomial with coefficients in $\mathbb{C}[t]$. Assuming

$$f(t(x+y), txy, txy^2, t^2y) = t^k y^j$$

and setting $y = 0$ gives $f(tx, 0, 0, 0) = 0$. Similarly setting $x = 0$ gives $f(ty, 0, 0, t^2y) = t^k y^j$. If $k \leq j$ it follows that necessarily $k = j$ and $f(x_1, 0, 0, x_2) = x_1$. Comparing with $f(tx, 0, 0, 0)$ we find $tx = 0$, a contradiction.

Claim 3. $\text{in}_> A$ is not finitely generated. Assuming $\text{in}_> A$ is finitely generated we can find a finite set σ_i of elements of A such that $\text{in}_> \sigma_i$ generate $\text{in}_> A$. By finiteness we can choose $m \gg 1$ such that for all i we have $\text{in}_> \sigma_i \neq t^{m-1}xy^m$. On the other hand by Claim 1 we know that for all m we have $t^{m-1}xy^m \in \text{in}_> A$. By the definition of a term ordering we know thus that $t^{m-1}xy^m$ must be a product of powers of initial terms of the elements σ_i . As x appears linearly it follows that there must be two generators σ_i, σ_j with $\text{in}_> \sigma_i = t^p xy^r$, respectively $\text{in}_> \sigma_j = t^q y^s$ with $p + q = m - 1$, $r + s = m$. By Claim 1 we must have $p \geq r - 1$ and by Claim 2 we must have $q > s$. Hence $p + q > r + s - 1 = m - 1$ so $p + q \geq m$, a contradiction. \square

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